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THE SOLUTION AND CONTINUALIZATION OF A WAVE PROBLEM FOR A STRING MESH†

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The problem of the propagation of a perturbation along a string mesh, formed by two systems of parallel strings with different characteristic impedances, is considered. An exact solution of the problem is obtained using established recurrence relations for orthogonal Kravchuk and Chebyshev–Hermite polynomials, and its continuous analogue is determined using limiting representations. The partial differential equations obtained are, in a limiting sense, equivalent to the initial equations in discrete variables. The structure of a mechanical system which is a continuous analogue of the string mesh is determined. Analogues of Green's functions are constructed in discrete and continuous versions. A comparison is made with the corresponding results at a physical level of rigour. In the case of an unbounded homogeneous mesh, formed by the two families of strings with a single transit time of the perturbation between the nodes, the function of the effect of the pulse which is applied to one of the nodes is obtained. This is identical to the square of the normalized Kravchuk polynomials apart from the specification of the arguments.

1. FORMULATION OF THE PROBLEM AND ITS SOLUTION BY CHANGING TO DISCRETE VARIABLES

Consider the problem of the transverse vibrations of a string mesh which occupies a half-plane and is formed by two systems of parallel equidistant strings with identical dynamic parameters, density ρ_i, tension T_i, propagation velocity of a perturbation $\upsilon_i = (T_i/\rho_i)^{1/2}$ within the limits of each system ($i = 1, 2$) and the same characteristic impedance $Z_i = (\rho_i T_i)^{1/2}$ for the strings of both systems ($Z_1 = Z_2$).

We choose an x axis which coincides with one of the strings of the horizontal system ($i = 1$) and a y axis which coincides with the last string of the second system ($i = 2$). We also agree to count the numbers n and m of strings of the first and second systems respectively from the origin of the coordinates. Then, for the internal points of the segments of the strings between the modal points, we have

$$\begin{aligned} &\frac{\partial^2 u_m}{\partial x^2} - \frac{1}{\upsilon_1^2}\frac{\partial^2 u_m}{\partial t^2} = 0, \quad nl_1 < x < (n+1)l_1, \quad n = 0, 1, 2, \dots \\ &\frac{\partial^2 w_n}{\partial y^2} - \frac{1}{\upsilon_2^2}\frac{\partial^2 w_n}{\partial t^2} = 0, \quad ml_2 < y < (m+1)l_2, \quad m = 0, \pm 1, \pm 2, \dots \end{aligned} \tag{1.1}$$

where u_m and w_n are displacements, t is the time and l_i is the length of the segments of the strings between the nodal points.

The formulation of the problem must be completed by the conditions at the nodes, that is, equality of the displacement of both strings and the dynamics equilibrium

$$u_m(nl_1, t) = w_n(ml_2, t) = U_n(t), \quad T_1\frac{\partial u_m}{\partial x}\bigg|_{x = nl_1 - 0}^{x = nl_1 + 0} + T_2\frac{\partial w_n}{\partial y}\bigg|_{y = ml_2 - 0}^{y = ml_2 + 0} = 0 \tag{1.2}$$

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as well as the initial and boundary conditions

$$u_m(x,0) = \frac{\partial u_m}{\partial t}(x,0) = 0, \quad w_n(y,0) = \frac{\partial w_n}{\partial t}(y,0) = 0$$
$$u_m(0,t) = w_0(y,t) = U_0(t) = 1 \tag{1.3}$$

It is obvious that the displacements at the nodal points $U_n(t)$ are independent of the number m, and their magnitudes at any three neighbouring nodal points of the horizontal system are found to be connected by a recurrence relations, which can be written in the uniform form

$$U_n(t) = \frac{1}{2}[U_{n-1}(t-\tau_1) + U_{n+1}(t-\tau_1)] +$$
$$+\frac{1}{2}[U_{n-1}(t-\tau_1-\tau_2) + U_{n+1}(t-\tau_1-\tau_2)] - U_n(t-2\tau_1-\tau_2) \quad \left(\tau_i = \frac{l_i}{v_i}\right) \tag{1.4}$$

if we agree to consider functions of the displacement with a negative value of the argument as being equal to zero. This relation can be obtained, for example, using an integral Laplace transformation in the solution of problem (1.1)–(1.3) and returning to the originals using a retardation theorem.

If we introduce an integral argument $N = [t/\tau_1]$, the number of nodal points n is considered as another discrete argument and the quantity $s = \tau_2/\tau_1$ is also assumed to be an integer, then Eq. (1.4) can be represented in the form

$$U(n,N) - \frac{1}{2}[U(n-1,N-1) + U(n+1,N-1)] =$$
$$= \frac{1}{2}[U(n-1,N-s-1) + U(n+1,N-s-1)] - U(n,N-s-2) \tag{1.5}$$

Hence, the initial problem (1.1)–(1.3) reduces to solving Eq. (1.5) in discrete variables with the conditions

$$U(n, N<0) = U(n>0,0) = 0, \quad U(0,N\geq 0) = 1 \tag{1.6}$$

We now define the required solution of problem (1.5) using the periods

$$U(n,N) = U_r(n,N) \quad \text{when} \quad r(s+1)-1 \leq N \leq (r+1)(s+1)-1, \quad r = 0, 1, 2 \ldots \tag{1.7}$$

For the functions U_r, we then obtain the equation

$$U_r(n,N) - \frac{1}{2}[U_r(n-1,N-1) + U_r(n+1,N-1)] =$$
$$= \frac{1}{2}[U_{r-1}(n-1,N-s-1) + U_{r-1}(n+1,N-s-1)] - U_{r-1}(n,N-s-2) \tag{1.8}$$

with the conditions

$$U_{-1}(n,N) = 0, \quad U_0(n>0,0) = 0, \quad U_r(0,N) = 1$$
$$U_r[n, r(s+1)-1] = U_{r-1}[n, r(s+1)-1], \quad r>0 \tag{1.9}$$

the last of which is the condition for the solutions for adjacent periods to match.

If we now introduce the functions $W_r(n, N)$, which are the solution of the same problem (1.8), (1.9) but for semi-infinite intervals of changes of the argument N,

$$N \geq r(s+1) - 1, r = 0, 1, 2, \ldots$$

then, obviously, the functions $W_r(n, N)$ will be identical to $U_r(n, N)$ and $U(n, N)$ in the finite intervals (1.7) and the required solution can be represented in terms of the difference between these functions $D_r(n, N) = W_r(n, N) - W_{r-1}(n, N)$, $r > 0$ by the formula

$$U(n,N) = W_0(n,N) + \sum_{r=1}^{[N/(s+1)]} D_r(n,N) \tag{1.10}$$

where the integer part of the number included in them is denoted by the square brackets.

The problem of determining the first term of series (1.10) has the form

$$W_0(n, N) - \frac{1}{2}[W_0(n-1, N-1) + W_0(n+1, N-1)] = 0,$$
$$W_0(n>0, 0) = 0, \quad W_0(0, N) = 1 \tag{1.11}$$

Since the equation in discrete variables connects the values of the required function at points with an even argument and, separately, with an odd sum of $n + N$ arguments, and the boundary and initial conditions for them are identical, the solution must consist of two identical parts with values of N which are shifted by unity with respect to one another. It is obvious that the terms of the series (1.1) can also be represented by the sum of two such terms

$$W_0(n, N) = R_0(n, N) + R_0(n, N-1), \quad D_r(n, N) = R_r(n, N) + R_r(n, N-1)$$

which are determined by the relations

$$R_r(n, N) - \frac{1}{2}[R_r(n-1, N-1) + R_r(n+1, N-1)] =$$
$$= \frac{1}{2}[R_{r-1}(n-1, N-s-1) + R_{r-1}(n+1, N-s-1)] - R_{r-1}(n, N-s-2)$$
$$R_r[n, r(s+1)-1] = 0, \quad R_r(0, N) = 0$$

If we now change to the new variable $N_r = N - rs$, then, for the function $V_r(n, N_r) = R_r(n, N)$, we obtain

$$V_r(n, N_r) - \frac{1}{2}[V_r(n-1, N_r-1) + V_r(n+1, N_r-1)] =$$
$$= \frac{1}{2}[V_{r-1}(n-1, N_r-1) + V_{r-1}(n+1, N_r-1)] - V_{r-1}(n, N_r-2) \tag{1.12}$$
$$V_r(n, r-1) = 0 \quad V_r(0, N_r) = 0$$

Since these relations are identical with the previously† established properties of normalized Kravchuk polynomials $K(r, s, N)$ when $p = q = 1/2$, $s = (n + N_r)/2$ (also, see Section 4), then

$$V_r(n, N_r) = \left(\frac{N_r - r + 1}{r}\right)^{1/2} K\left(r-1, \frac{n+N_r}{2}, N_r\right) K\left(r, \frac{n+N_r}{2}, N_r\right), \quad r>0$$

The solution of problem (1.1) is conveniently sought in the form

$$W_0(n, N) = \sum_{k=0}^{N} F(n, k)$$

where the function $F(n, N)$ is the solution of the same Eq. (1.11) with the conditions

$$F(n, 0) = 0, \quad F(0, N) = \begin{cases} 1 & \text{when } N = 0 \\ 0 & \text{when } N > 0 \end{cases}$$

and, in turn, can be represented when $n \geq 0, N \geq 1$ in terms of the discrete analogue of Green's function

$$F(n, N) = \frac{1}{2}G_0(n-1, N-1) - \frac{1}{2}G_0(n+1, N-1), \quad G_0(n, N) = \left(\frac{1}{2}\right)^N C_N^{(n+N)/2}$$

The correctness of this equality follows from the fact that it is obviously satisfied in the special cases when $n \geq 0$, $N = 1$ and $n = 0$, $N \geq 1$.

† Volkov, I. A., Modelling of the processes of piezo- and thermal conduction in binary media related to problems in the oscillation theory. Doctorate dissertation, 01.02.05, 01.02.04, Moscow, 1990.

After simplifications, we find that

$$F(n, N) = \frac{n}{N} C_N^{(n+N)/2} \left(\frac{1}{2}\right)^N$$

Hence, the solution of problem (1.5), (1.6) has the form

$$U(n, N) = W_0(n, N) + \sum_{r=1}^{[N/s]} [V_r(n, N-rs) + V_r(n, N-rs-1)]$$

$$W_0(n, N) = \begin{cases} 1 & \text{when} \quad n = 0 \\ \sum_{r=n}^{N} \frac{n}{r} C_r^{(n+r)/2} \left(\frac{1}{2}\right)^r & \text{when} \quad n > 0 \end{cases} \tag{1.13}$$

$$V_r(n, N) = ((N-r+1)/r)^{1/2} K(r-1, (n+N)/2, N) K(r, (n+N)/2, N)$$

By the quantities $C_k^{(n+r)/2}$ and $K(r, (n + N)/2, N)$, we mean binomial coefficients and normalized Kravchuk polynomials in the domain of their standard definition and zeros in the remaining cases.

Using the same technique, another solution of Eq. (1.5), an analogue of Green's functions, can be constructed

$$G_2(n, n_0, N) = G_1(n, n_0, N) + G_1(n, n_0 - 1, N)$$

$$G_1(n, n_0, N) = B_0^2(n - n_0, N) - \sum_{r=1}^{[N/s]} [B_{r-1}^2(n - n_0, N - rs - 1) - B_r^2(n - n_0, N - rs)] \tag{1.14}$$

$$B_r^2(n, N) = K^2(r, (n+N)/2, N)$$

which corresponds to the conditions

$$\sum_{(n)} G_2(n, n_0, N) = 2, \quad G_2(n, n_0, 0) = \begin{cases} 1 & \text{when} \quad n = n_0, n_0 - 1 \\ 0 & \text{when} \quad n \neq n_0, n_0 - 1 \end{cases}$$

2. CONTINUOUS ANALOGUE OF THE PROBLEM IN DISCRETE VARIABLES

It is well known [1] that the limiting relation

$$\lim_{N \to \infty} (2pqN)^{1/4} K(r, s, N) = \Psi_r(z), \quad z = \frac{s - pN}{\sqrt{2pqN}} \tag{2.1}$$

exists between normalized Kravchuk polynomials $K(r, s, N)$ and Chebyshev–Hermite $\Psi_r(z)$ polynomials for fixed r and z.

Starting from this and the integral de Moivre–Laplace theorem, it is possible to find limiting representations of the solutions (1.13) and (1.14) which have been presented. The first of them, for example, takes the form

$$u(\xi, \theta) = \operatorname{erfc}\left(\frac{\xi}{\sqrt{2\theta}}\right) + \sum_{r=1}^{[\theta/\tau]} \sqrt{\frac{2}{r}} \Psi_{r-1}\left(\frac{\xi}{\sqrt{2(\theta - r\tau)}}\right) \Psi_r\left(\frac{\xi}{\sqrt{2(\theta - r\tau)}}\right) \tag{2.2}$$

where

$$\xi = \frac{x}{l_1}, \quad \theta = \frac{v_1 t}{l_1}, \quad \tau = \frac{v_1 l_2}{v_2 l_1} \tag{2.3}$$

We will show that expression (2.2) is a solution of the differential equation that is the continuous analogue of Eq. (1.5). For this purpose, we will first prove that the functions

$$F_r(z) = \left(\frac{2}{r}\right)^{1/2} \Psi_{r-1}(z)\Psi_r(z), \quad r>0 \tag{2.4}$$

satisfy the relation

$$\frac{d^2}{dz^2}[F_r(z)+F_{r-1}(z)] + 2z\frac{d}{dz}[F_r(z)-F_{r-1}(z)] = 0 \tag{2.5}$$

We shall start from the well-known equation [2]

$$\Psi_r''(z) + (2r+1-z^2)\Psi_r(z) = 0 \tag{2.6}$$

and the equalities

$$\begin{aligned} \left(\frac{r+1}{2}\right)^{1/2}\Psi_{r+1}(z) + \left(\frac{r}{2}\right)^{1/2}\Psi_{r-1}(z) &= z\Psi_r(z), \\ \left(\frac{r+1}{2}\right)^{1/2}\Psi_{r+1}(z) - \left(\frac{r}{2}\right)^{1/2}\Psi_{r-1}(z) &= -\Psi_r'(z) \end{aligned} \tag{2.7}$$

Substituting expression (2.4) into Eq. (2.5), after some simple transformations and simplifications using formulae (2.7), we obtain the equality

$$\Psi_r[\Psi_r'' + (2r+1-z^2)\Psi_r]' + 3\Psi_r'[\Psi_r'' + (2r+1-z^2)\Psi_r] = 0 \tag{2.8}$$

the identical satisfaction of which, by virtue of relation (2.6), is obvious. If, additionally, it is assumed that

$$F_0(z) = \operatorname{erfc}(z) \tag{2.9}$$

then it can be shown by direct verification that $F_0(z)$ and $F_1(z)$ are related by the same equality (2.5).

We will now consider the boundary-value problem

$$\begin{aligned} &\frac{1}{2}\frac{\partial^2}{\partial\xi^2}[u(\xi,\theta)+u(\xi,\theta-\tau)] = \frac{\partial}{\partial\theta}[u(\xi,\theta)-u(\xi,\theta-\tau)] \\ &u(\xi,\theta\le 0) = 0, \quad u(0,\theta\ge 0) = 1 \end{aligned} \tag{2.10}$$

By analogy with problem (1.5), (1.6), we determine the required function with respect to the periods

$$u(\xi,\theta) = u_r(\xi,\theta) \quad \text{when} \quad r\tau \le \theta < (r+1)\tau, \quad r = 0,1,2,\dots$$

where the function $u_r(\xi, \theta)$ correspond to the relations

$$\begin{aligned} &\frac{1}{2}\frac{\partial^2}{\partial\xi^2}[u_r(\xi,\theta)+u_{r-1}(\xi,\theta-\tau)] = \frac{\partial}{\partial\theta}[u_r(\xi,\theta)-u_{r-1}(\xi,\theta-\tau)] \\ &u_{-1}(\xi,\theta) = 0, \quad u_0(\xi,0) = 0, \quad \xi\ne 0; \quad u_r(\xi,r\tau) = u_{r-1}(\xi,r\tau), \\ &u_r(0,\theta) = 1, \quad r = 0,1,2,\dots \end{aligned} \tag{2.11}$$

We now introduce the functions $\omega_r(\xi, \theta)$ which are a solution of the same problem (2.11) but in semi-bounded intervals of variation of the argument θ

$$\theta \ge r\tau, \quad r = 0,1,2,\dots$$

The required solution will then be represented in terms of the different of these functions

$$u(\xi,\theta) = \omega_0(\xi,\theta) + \sum_{r=1}^{[\theta/\tau]} \delta_r(\xi,\theta); \quad \delta_r(\xi,\theta) = \omega_r(\xi,\theta) - \omega_{r-1}(\xi,\theta), \quad r>0 \tag{2.12}$$

The first term in equality (2.12) is the well-known self-similar solution of the heat conduction equation

$$\omega_0(\xi,\theta) = \delta_0(\xi,\theta) = \operatorname{erfc}\left(\frac{\xi}{\sqrt{2\theta}}\right)$$

and the functions under the summation sign satisfy relations which, after the change of variables

$$\theta = \theta_r + r\tau, \quad \delta_r(\xi,\theta) = f_r(\xi,\theta_r)$$

can be written in the form

$$\frac{1}{2}\frac{\partial^2}{\partial\xi^2}[f_r(\xi,\theta_r) + f_{r-1}(\xi,\theta_r)] = \frac{\partial}{\partial\theta_r}[f_r(\xi,\theta_r) - f_{r-1}(\xi,\theta_r)] \tag{2.13}$$

$$f_r(\xi,\theta_r) = 0, \quad f_r(0,\theta_r) = 0, \quad r = 1, 2, 3, \dots \tag{2.14}$$

The self-similar functions

$$f_r(\xi,\theta_e) = \left(\frac{2}{r}\right)^{1/2}\Psi_{r-1}(\xi(2\theta_r)^{-1/2})\Psi_r(\xi(2\theta_r)^{-1/2}) = F_r(\xi(2\theta_r)^{-1/2}) \tag{2.15}$$

which are identical with the functions (2.4) apart from the notation, satisfy problem (2.13), (2.14). In fact, their direct substitution into Eq. (2.13) reduces it to the form (2.5), and it follows from the properties of Chebyshev–Hermite polynomials

$$\Psi_r(\infty) = 0, \quad \Psi_{2r+1}(0) = 0, \quad r = 0, 1, 2, \dots$$

that conditions (2.14) are satisfied

Comparing the results, we finally obtain the solution of problem (2.10) in a form which is identical with expression (2.2) whence Green's function

$$\begin{aligned} G(\xi,\xi_0,\theta) &= \frac{1}{2}\frac{\partial}{\partial\xi_0}u(\xi-\xi_0,0) = \frac{1}{\sqrt{2\pi\theta}}\exp\left[\frac{(\xi-\xi_0)^2}{2\theta}\right] - \\ &- \sum_{r=1}^{[\theta/\tau]} \frac{1}{\sqrt{2(\theta-r\tau)}}\left\{\Psi_{r-1}^2\left[\frac{\xi-\xi_0}{\sqrt{2(\theta-r\tau)}}\right] - \Psi_r^2\left[\frac{\xi-\xi_0}{\sqrt{2(\theta-r\tau)}}\right]\right\} \end{aligned} \tag{2.16}$$

$$\int_{-\infty}^{+\infty} G(\xi,\xi_0,\theta)d\xi = 1$$

can also be found using the standard method.

When $\theta < \tau$, the right-hand side of equality (2.16) degenerates into the source function for the heat conduction equation. The second equality holds by virtue of the orthonomality of Chebyshev–Hermite functions.

3. THE STRUCTURE OF A MECHANICAL SYSTEM CORRESPONDING TO THE CONTINUAL EQUATION

We will show that an equation of the form (2.10) can be interpreted as the equation of the transverse vibrations of a string-membrane system with zero values of certain dynamic parameters.

We will write the condition for the dynamic equilibrium of an inertialess ($\rho_1 = 0$) string joined to a membrane as

$$T_1 \frac{\partial^2 u_1}{\partial x^2} + T_2' \frac{\partial u_2}{\partial y}\bigg|_{y=-0}^{y=+0} = 0 \tag{3.1}$$

where the tensile stress in the membrane per unit length of the string is denoted by a prime, and, also, the equations of the vibrations of a membrane which does not possess elastic properties in the longitudinal direction (along the x axis),

$$\frac{\partial^2 u_2}{\partial y^2} - \frac{1}{v_2^2}\frac{\partial^2 u_2}{\partial t^2} = 0, \quad v_2 = \sqrt{\frac{T_2'}{\rho_2'}} \tag{3.2}$$

and the condition for the displacements of the membrane and the string to be matched

$$u_2(y = 0, t; x) = u_1(x, t) \tag{3.3}$$

In the case of a membrane, associated with a single string which is unconstrained along the y axis and unperturbed at the initial instant of time, the displacement function can be represented as the solution of problem (3.2)–(3.4) and its substitution into (3.1) leads to the heat conduction equation.

$$u_2(y, 0; x) = \frac{\partial u_2}{\partial y}(y, 0; x) = 0 \tag{3.4}$$

If, however, the membrane has the form of a strip of width l_2, connected to a centrally located string, then, for the boundary conditions

$$\frac{\partial u_2}{\partial y}\bigg|_{y=\pm l_2/2} = 0 \tag{3.5}$$

the procedure of eliminating the unknown u_2 from system (3.1)–(3.5) leads, as can be shown, to the equation

$$\frac{\partial^2}{\partial x^2}\left[u_1(x,t) + u_1\left(x, t - \frac{l_2}{v_2}\right)\right] = \frac{1}{a^2}\frac{\partial}{\partial t}\left[u_1(x,t) - u_1\left(x, t - \frac{l_2}{v_2}\right)\right], \quad a^2 = \frac{T_1}{2\sqrt{\rho_2' T_2'}} \tag{3.6}$$

which is identical with the continual equation (2.10) when account is taken of the notation used in (2.3), the equality of the impedances of all of strings forming the mesh and the additional condition

$$\rho_2' = \rho_2/l_1; \quad T_2' = T_2/l_1 \tag{3.7}$$

and the matching of the dynamic parameters of the membrane and the vertical strings in the string mesh.

By comparing the structures of the initial string mesh with the string-membrane system as its continuous analogue, condition (3.7) may appear to be obvious. It is natural to treat a membrane which does not possess elasticity along the x axis as segments of vertical strings continuously distributed ("smeared out") along the x axis. In fact, the replacement of a discrete element by a continuously distributed element is a widespread method of continualization at a physical level of rigour. However, there is a substantial difference between the results obtained using the exact and the approximate approaches. In the first case, the strings are inertialess in a continuous system while, in the second case, they have the density of the strings of the initial system. At the same time, an additional terms must appear in Eq. (3.1) which complicates the resultant equation (3.6), the solutions (2.2) and (2.18), which are simple in form, no longer hold good, and the search for new solutions (which are more complex and less rigorous) becomes problematical.

We note that the orthogonality of the two systems of strings forming the mesh has not been used anywhere in the solution of the problems, so that the results obtained also hold for any non-zero angle of their intersection (in oblique-angled coordinates x and y). Moreover, another location of the boundary is also possible. Apart from its coincidence with one of the strings of the vertical system ($x = 0$), practically the same results can also be obtained in the case of a diagonal arrangement with respect to the cells of the mesh.

4. APPLICATION. A RECURRENCE RELATION FOR THE SQUARES OF NORMALIZED KRAVCHUK POLYNOMIALS

The classical orthogonal Kravchuk polynomials (of a discrete variable) are defined [1] by the generating function

$$\sum_{r=0}^{N} k_r(s, N)z^r = (1+qz)^s(1-pz)^{N-s}, \quad N \geq s \geq 0, \quad p > 0, \quad q = 1-p > 0 \tag{4.1}$$

We now introduce normalized Kravchuk polynomials into the treatment, which we subsequently agree to call Kravchuk functions

$$K(r, s, N) = (C_N^s/C_N^r)^{1/2} p^{(s-r)/2} q^{(N-s-r)/2} k_r(s, N) \tag{4.2}$$

where the quantities C_N^s and C_N^r are binomial coefficients.

We will now show that the squares of these functions satisfy the recurrence relation

$$\begin{aligned} &K^2(r, s, N+1) - qK^2(r, s, N) - pK^2(r, s-1, N) = pK^2(r-1, s, N) + \\ &+ qK^2(r-1, s-1, N) - K^2(r-1, s-1, N-1) \end{aligned} \tag{4.3}$$

For the proof, we use the equalities

$$\begin{aligned} &[p(s+1)]^{1/2}K(r, s+1, N+1) + [q(N-s+1)]^{1/2}K(r, s, N+1) = \\ &= (N-r+1)^{1/2}K(r, s, N) \end{aligned} \tag{4.4}$$

$$q[(s+1)]^{1/2}K(r, s+1, N+1) - [p(N-s+1)]^{1/2}K(r, s, N+1) = r^{1/2}K(r-1, s, N) \tag{4.5}$$

$$(ps)^{1/2}K(r, s-1, N-1) + [q(N-s)]^{1/2}K(r, s, N-1) = (N-r)^{1/2}K(r, s, N) \tag{4.6}$$

$$(qs)^{1/2}K(r, s-1, N-1) - [p(N-s)]^{1/2}K(r, s, N-1) = (r+1)^{1/2}K(r+1, s, N) \tag{4.7}$$

$$\begin{aligned} &[pq(r+1)(N-r)]^{1/2}K(r+1, s, N) + [pN + (q-p)r - s]K(r, s, N) + \\ &+ [pqr(N-r+1)]^{1/2}K(r-1, s, N) = 0 \end{aligned} \tag{4.8}$$

the first two of which are established directly form relations (4.1) and (4.2), and the last two are established using expression (4.2) and the formula which is obtained on differentiating the generating function with respect to z. Equality (4.8) corresponds to the relation between three Kravchuk polynomials with consecutive indices.

On multiplying the right-hand and left-hand sides of equalities (4.4) and (4.5), we find

$$\begin{aligned} &[pqr(N-r+1)]^{1/2}K(r-1, s, N)K(r, s, N) = \\ &= pq(s+1)K^2(r, s+1, N+1) - pq(N-s+1)K^2(r, s, N+1) + \\ &+ (q-p)[pq(s+1)(N-s+1)]^{1/2}K(r, s, N+1)K(r, s+1, N+1) \end{aligned} \tag{4.9}$$

The last term in this relation can be expressed in terms of the squares of Kravchuk functions by two methods: directly from the relation obtained when both sides of equality (4.4) are squared

$$\begin{aligned} &2[pq(s+1)(N-s+1)]^{1/2}K(r, s, N+1)K(r, s+1, N+1) = (N-r+1)K^2(r, s, N) - \\ &- q(N-s+1)K^2(r, s, N-1) - p(s+1)K^2(r, s+1, N+1) \end{aligned} \tag{4.10}$$

and after subtraction of the transformed equalities (4.4) and (4.5) in a similar manner

$$2[pq(s+1)(N-s+1)]^{1/2}K(r,s,N+1)K(r,s+1,N+1) = q(s+1)K^2(r,s+1,N+1)+ \\ +p(N-s+1)K^2(r,s,N+1)-rK^2(r-1,s,N) \tag{4.11}$$

Substitution of expressions (4.10) and (4.11) into relation (4.9) leads to the formulae

$$[pqr(N-r+1)]^{1/2}K(r-1,s,N)K(r,s,N) = \frac{1}{2}p(s+1)K^2(r,s+1,N+1)- \\ -\frac{1}{2}q(N-s+1)K^2(r,s,N-1)+\frac{1}{2}(q-p)(N-r+1)K^2(r,s,N) \tag{4.12}$$

$$[pq(r+1)(N-r)]^{1/2}K(r,s,N)K(r+1,s,N) = \frac{1}{2}q(s+1)K^2(r+1,s+1,N+1)- \\ -\frac{1}{2}p(N-s+1)K^2(r+1,s,N+1)-\frac{1}{2}(q-p)(r+1)K^2(r+1,s,N) \tag{4.13}$$

If expressions (4.12) and (4.13) for its boundary terms are substituted into equality (4.8), which has been multiplied term by term by $K(r, s, N)$, we arrive at the relation

$$(N-2s-1)K^2(r,s,N-1)+p[(s+1)K^2(r,s+1,N)-(N-s)K^2(r+1,s,N)]- \\ -q[(N-s)K^2(r,s,N)-(s+1)K^2(r+1,s+1,N)] = 0 \tag{4.14}$$

which only contains squares of Kravchuk functions.

In exactly the same way as relation (4.14) was obtained from equalities (4.4), (4.5) and (4.8), we can obtain

$$(N-2s-1)K^2(r+1,s+1,N+1)+p[(s+1)K^2(r+1,s,N)-(N-s)K^2(r,s+1,N)]- \\ -q[(N-s)K^2(r+1,s+1,N)-(s+1)K^2(r,s,N)] = 0 \tag{4.15}$$

from (4.6)–(4.8) and by the addition of expressions (4.14) and (4.15), after contraction by the common factor $(N - 2s - 1)$, we can obtain a recurrence relation of the required form (4.3).

In a similar way, it is possible to prove that the functions

$$F(r,s,N) = 2\left(pq\frac{N-r+1}{r}\right)^{1/2}K(r-1,s,N)K(r,s,N), \quad r>0 \tag{4.16}$$

also satisfy a relation of the same form.

As an example of the use of relation (4.3) which has been proved, we find the functions for the effect of a pulse applied to one of the modal points of an unbounded homogeneous mesh which is formed by two families of strings with a single perturbation transit time between the nodal points $\tau = l_1/\upsilon_1 = l_2/\upsilon_2$.

Under the conditions adopted, the problem reduces to solving an equation for the displacements at the nodal points

$$U(n,m,N+1)+U(n,m,N-1) = \alpha_1[U(n+1,m,N)+U(n-1,m,N)]+ \\ +\alpha_2[U(n,m+1,N)+U(n,m-1,N)] \tag{4.17}$$

with the conditions

$$U(n,m,N<0) = 0, \quad U(0,0,0) = U(0,0,1)$$

where

$$n = [x/l_1], \quad m = [y/l_2], \quad N = [t/\tau], \quad \alpha_i = Z_i/(Z_1+Z_2), \quad i = 1,2$$

Here, n, m and N are dimensionless integer-valued coordinates (is the numbers of the nodal points) and the time, the origin of which ($n = m = N = 0$) is a point and the instant of application of the pulse, while the coefficients α_i are determined by the ratio of the impedances Z of the strings of the two systems.

Equation (4.17) is identical with recurrence relation (4.3) for the squares of Kravchuk functions apart from the notation

$$r = (N-m-n)/2, \quad s = (N-m+n)/2, \quad p = \alpha_1, \quad q = \alpha_2$$

and, if without loss in generality in the problem, it is assumed that the initial displacement at the point of application of the pulse is a unit displacement: $U(0, 0, 0) = K^2(0, 0, 0) = 1$, then, for the function for the effect of the pulse, we obtain the simple analytical expression

$$U(m, n, N) = K^2\left(\frac{N-m-n}{2}, \frac{N-m+n}{2}, N\right) + K^2\left(\frac{N-m-n-1}{2}, \frac{N-m+n-1}{2}, N-1\right)$$

where one of the terms (with half-integer values of the arguments of Kravchuk functions) is clearly equal to zero.

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